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Iterative methods for solving extended general mixed variational inequalities

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ABSTRACT

In this paper, we introduce and consider a new class of mixed variational inequalities involving four operators, which are called extended general mixed variational inequalities. Using the resolvent operator technique, we establish the equivalence between the extended general mixed variational inequalities and fixed point problems as well as resolvent equations. We use this alternative equivalent formulation to suggest and analyze some iterative methods for solving general mixed variational inequalities. We study the convergence criteria for the suggested iterative methods under suitable conditions. Our methods of proof are very simple as compared with other techniques. The results proved in this paper may be viewed as refinements and important generalizations of the previous known results.

1. Introduction

The variational principle, the origin of which can be traced back to Fermat, Newton, Leibniz, Bernouli, Euler and Lagrange, has been one of the major principles of mathematical and engineering sciences for more than two centuries. It can be used to interpret the basic principles of mathematical and physical sciences in forms of simplicity and elegance. The variational principles have played a fundamental and important part as a unifying influence in the sciences and have played a fundamental role in the development of the general theory of relativity, gauge field theory in modern particle physics and soliton theory. In recent years, these variational principles have been enriched by the discovery of the variational inequalities theory, which is mainly due to Stampacchia [1] (from 1964). Variational inequalities theory constituted a significant and novel extension of the variational principles and describes a broad spectrum of interesting and fascinating developments involving a link among various fields of mathematics, physics, economics, equilibrium, financial, optimization, regional and engineering sciences. In fact, it has been shown that variational inequalities theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of problems. It is well known that the variational inequality theory is related to the simple fact that the minimum u of a differentiable convex function F on a convex set in any normed space can be characterized by an inequality of the type $\langle F'(u), v-u\rangle \geq 0,\ \forall u, v \in K$, where $F'(u)$ is the differential of F at $u \in K$. However, it is amazing that this theory allows many diversified applications in various branches of pure and applied sciences. We would like to point out that the variational inequalities theory can be regarded as a natural development of the 19th and 20th problems of Hilbert, which he formulated in his famous Paris lecture in 1900; see, for example, [2] for more details.

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Convexity has played an important role in the development of the variational inequalities theory and has been generalized in several directions. A significant generalization of the convex set is the introduction of the *gh*-convex set [3] and the *gh*-convex function [4–8]. It has been shown [4–8] that these nonconvex functions enjoy some nice properties which convex functions have. We would like to emphasize that the *gh*-convex set and *gh*-convex functions may not be convex sets and convex functions. In this paper, we show that the minimum of the sum of a differentiable nonconvex function and a nondifferentiable nonconvex function on the nonconvex set can be characterized by a class of variational inequalities. This result (Lemma 2.1) generalizes the corresponding known result for the convex functions. This result also inspired and motivated us to introduce a new class of variational inequalities, which are called extended general mixed variational inequalities [9]; see (2.1). We note that if the nonlinear term involving the extended general mixed variational inequalities is the indicator function of a closed convex set, then the general mixed variational inequalities are exactly the extended general variational inequalities, considered by Noor [4–8]. One can show that the variational inequality introduced by Stampacchia [1] is a special case of the extended general mixed variational inequality (2.1). This clearly shows that the notion of the general mixed variational inequality is quite a general and unifying one.

In recent years, several numerical methods including projection and its variant forms, Wiener–Hopf equations and auxiliary principle techniques have been developed. This class of iterative methods have witnessed great progress in recent years. Apart from theoretical interest, the main advantage of these methods, which makes them successful in addressing real world problems, is in the computation. These methods have the ability to handle large-size problems of dimensions for which other methods cease to be efficient. In brief, the field of the iterative method itself is vast; see [10–23,2,24–32,4–9, 33–40,42]. It is well known that for mixed variational inequalities involving the nonlinear terms, one cannot use projection and its variant form to establish the equivalence between the general mixed variational inequalities and the fixed point problem. To overcome this drawback, we assume that the nonlinear term involving general mixed variational inequalities is proper, convex and lower semicontinuous. In this case, it is known that the subdifferential of a proper, convex and lower semicontinuous form is a maximal monotone operator. This characterization enables us to define the resolvent operator associated with the maximal monotone operator. We use the resolvent operator technique to establish the equivalence between the general mixed variational inequalities and the fixed point problem, which is Lemma 3.1. The novel feature of the technique is that the resolvent step involves the subdifferential of a proper, convex and lower semicontinuous function part only and the other part facilitates the problem decomposition. This can lead to the development of very efficient methods, since one can treat each part of the original operator independently. We use this alternative formulation to study the existence of a solution of the general mixed variational inequality, which extends the known result. This equivalent formulation is used to suggest and analyze a new Mann-type iterative method for solving the general mixed variational inequalities; see Algorithm 3.1. In the process of proving the main results (Theorems 3.1 and 3.2), we use the resolvent operator technique. We note that if the proper, convex and lower semicontinuous function in the general mixed variational inequalities is an indicator function of a closed convex set K in the real Hilbert space, then the resolvent operator is exactly the operator of projection of H onto the closed convex set K. In this case, we recover the algorithm of Noor [7] for solving the extended general variational inequalities.

Related to the general mixed variational inequalities, we have the problem of solving the resolvent equations, the origin of which can be traced back to Noor [22]. Using again the resolvent operator technique, we establish the equivalence between the extended general mixed variational inequalities and the resolvent equations. Here, we would like to emphasize the fact that, if the nonlinear term in the general mixed variational inequalities is an indicator function of a convex closed subset in the Hilbert space, then the resolvent operator is exactly the operator of projection from the Hilbert space onto the closed convex set and consequently the resolvent equations are equivalent to the Wiener–Hopf equations, which were initially introduced by Shi [40]. It turned out that this approach is more general and flexible. In Section 4, we use the resolvent equations technique to suggest and analyze a number of new iterative methods for solving the general mixed variational inequalities and related optimization problems. We prove the strong convergence (the main theorem, Theorem 4.1) of the new iterative method under same conditions as for Theorem 3.2. Since the extended general mixed variational inequalities include the various classes of variational inequalities and nonlinear programming problems as special cases, the results obtained in this paper continue to hold for these problems. The ideas and techniques of this paper may be a starting point for a wide range of novel and innovative applications in various fields.

2. Preliminaries

Let K be a nonempty closed and convex set in a real Hilbert space H, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let $T : H \longrightarrow H$ be a nonlinear operator and S be a nonexpansive operator. Let P_K be the projection of H onto the convex set K. Let $\varphi : H \longrightarrow R \cup \{\infty\}$ be a continuous function.

For given nonlinear operators $T, g, h : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, h(v) - g(u)\rangle + \varphi(h(v)) - \varphi(g(u)) \geq 0, \quad \forall v \in H. \tag{2.1}$$

An inequality of type (2.1) is called the *extended general mixed variational inequality involving four operators* and is quite different than all other classes of variational inequalities. Extended general mixed variational inequalities were introduced by Noor [9]. A wide class of problems arising in pure and applied sciences can be studied via the extended general mixed variational inequalities (2.1).

Example 2.1 (*[9]*). As an application of problem (2.1), we show that the optimality condition for the minimum of a sum of differentiable and nondifferentiable nonconvex functions on a nonconvex set K in H can be characterized by the general mixed variational inequality of type (2.1). This result is due to Noor [9]. We include all the details to convey an idea of the technique.

For this purpose, we recall the following well known concepts; see [3].

Definition 2.1. Let K be any set in H. The set K is said to be gh-convex if there exist functions $g, h : H \longrightarrow H$ such that

$$h(u) + t(g(v) - h(u)) \in K, \quad \forall u, v \in H : h(u), g(v) \in K,\ t \in [0, 1].$$

Note that every convex set is gh-convex, but the converse is not true; see [3]. If $g = h = I$, then the gh-convex set K is called the convex set.

Definition 2.2. The function $F : K \longrightarrow H$ is said to be gh-convex if there exist functions g, h such that

$$F(g(u) + t(h(v) - g(u))) \leq (1 - t)F(g(u)) + tF(h(v)), \quad \forall u, v \in H : h(u), g(v) \in K,\ t \in [0, 1].$$

Clearly every convex function is gh-convex, but the converse is not true. For the properties and various classes of the gh-convex functions, see [7,8]. We note that if the gh-convex function is differentiable, then

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle, \quad u, v \in H : h(u), g(v) \in K,$$

and conversely.

For a given differentiable gh-convex function F and a nondifferentiable gh-convex function φ, we consider a functional of the type

$$I[v] = F(v) + \varphi(v), \quad \forall v \in K. \tag{2.2}$$

One can prove that the minimum of the functional $I[v]$ on the gh-convex set K can be characterized by a class of variational inequalities (2.1). For the sake of completeness and to convey an idea of the technique, we include the proof.

Lemma 2.1. *Let F be a differentiable gh-convex function and φ be a nondifferentiable gh-convex function on the gh-convex set K. Then $u \in K$ is the minimum of $I[v]$, defined by (2.2), on $K \subset g(H)$, if and only if $u \in H : g(u) \in K$ satisfies the inequality*

$$\langle F'(g(u)), h(v) - g(u)\rangle + \varphi(h(v)) - \varphi(g(u)) \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.3}$$

where $F'(g(u))$ is the differential of F at $g(u) \in K$.

Proof. Let $u \in H : g(u) \in K$ be a minimum of functional $I[v]$ on K. Then

$$I[g(u)] \leq I[g(v)], \quad \forall v \in H : g(v) \in K. \tag{2.4}$$

K is a gh-convex set; so, for all $u, v \in H : g(u), h(v) \in K$, $t \in [0, 1]$, $g(v_t) = g(u)+t(h(v)-g(u)) \in K$. Setting $g(v) = g(v_t)$ in (2.4), we have

$$I[g(u)] \leq I[g(u) + t(h(v) - g(u))],$$

which implies that

$$\begin{aligned} F(g(u)) + \varphi(g(u)) &\leq F(g(u) + t(h(v) - g(u))) + \varphi(g(u) + t(h(v) - g(u))) \\ &\leq F(g(u) + t(h(v) - g(u))) + \varphi(g(u)) + t[\varphi(h(v)) - \varphi(g(u))]. \end{aligned}$$

From this, we have

$$F(g(u) + t(h(v) - g(u))) - F(g(u)) + t(\varphi(h(v)) - \varphi(g(u))) \geq 0, \quad \forall v \in K.$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(g(u)), h(v) - g(u)\rangle + \varphi(h(v)) - \varphi(g(u)) \geq 0, \quad \forall v \in H : h(v) \in K,$$

which is the required result (2.3).

Conversely, let $u \in K$ satisfy the inequality (2.3). Since F is a g-convex function, $\forall u, v \in H : g(u), h(v) \in K,\ t \in [0, 1],\ g(u) + t(h(v) - g(u)) \in K$.

Consider

$$\begin{aligned} I[g(u)] - I[g(v)] &= F(g(u)) + \varphi(g(u)) - F(h(v)) + \varphi(h(v)) \\ &\leq \langle F'(g(u)), g(u) - h(v)\rangle + \varphi(h(v)) - \varphi(g(u)) \\ &\leq 0, \quad \text{using (2.3)}, \end{aligned}$$

which implies that

$$I[g(u)] \leq I[g(v)], \quad \forall v \in H : g(v) \in K$$

showing that $u \in H : g(u) \in KK$ is the minimum of $I[v]$ on K in H.

Lemma 2.1 implies that gh-convex programming problems can be studied via the general mixed variational inequality (2.1) with $Tu = F'(g(u))$.

If the nondifferentiable function $\varphi \equiv 0$, then Lemma 2.1 reduces to the following result, which is due to Noor [7]. $\square$

Lemma 2.2. *Let F be a differentiable g-convex function. Then $u \in K$ is the minimum of F on $K \subset g(H)$ if and only if $u \in K$ satisfies the inequality*

$$\langle F'(g(u)), h(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

where $F'(g(u))$ is the differential of F at $u \in H : g(u) \in K$.

We would like to point out that the general mixed variational inequality (2.1) can be written in an equivalent form as follows: Find $u \in H$ such that

$$\langle \rho Tu + h(u) - g(u), h(v) - g(u)\rangle + \rho\varphi(h(v)) - \rho\varphi(g(u)) \geq 0, \quad \forall v \in H. \tag{2.5}$$

This equivalent formulation plays an important part in developing iterative methods for solving the general mixed variational inequalities.

If $g = I$, the identity operator, then the general mixed variational inequality problems (2.1) and (2.5) are equivalent to that of finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H, \tag{2.6}$$

which is known as the mixed variational inequality or the variational inequality of the second type. We note that if the function φ in the general mixed variational inequality (2.5) is proper, convex and lower semicontinuous, then problem (2.5) is equivalent to that of finding $u \in H$ such that

$$0 \in Tu + h(u) - g(u) + \partial\varphi(g(u)), \tag{2.7}$$

which is known as the problem of finding a zero of a sum of two (or more) monotone operators. It is well known that a large class of problems arising in industry, ecology, finance, economics, transportation, network analysis and optimization can be formulated and studied in the framework of (2.1) and (2.7); see [1–42].

If φ is an indicator function of a closed convex set K in H, that is,

$$\varphi(u) = I_K(v) = \begin{cases} 0, & \text{if } v \in K; \\ +\infty, & \text{otherwise}, \end{cases}$$

then the general mixed variational inequality problem (2.1) is equivalent to that of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, h(v) - g(u)\rangle \geq 0, \quad \forall v \in H : h(v) \in K, \tag{2.8}$$

which is called a extended general variational inequality, introduced and studied by Noor [4–9]. From Lemma 2.2, we see that the minimum of a class of differentiable nonconvex functions on the nonconvex set can be characterized by extended general variational inequalities of the type (2.8). For applications, numerical methods and other aspects of the extended general variational inequalities (2.8), see [14,15,4–9].

We note that for $h(u) = g(u)$, the inequality problem (2.8) is equivalent to that of finding $u \in H : g(u) \in K$ such that

$$\langle T_1(u), g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.9}$$

with $T_1(u) \equiv T(h^{-1}(g(u)))$, which is known as the *general variational inequality* and has been studied extensively in recent years. For the formulation, numerical methods, sensitivity analysis and other aspects of the general variational inequalities, see [18–23,2,24–32,34–37].

If $g = h = I$, then problems (2.9) and (2.8) reduce to that of finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.10}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [1] in 1964. For the numerical methods, formulations and applications of the mixed variational inequalities, readers may consult the recent state-of-the-art papers [41,3,10–23,2,24–32,4–9,33–40,42] and the references therein.

We now recall some well known concepts and results.

Definition 2.3 (*[41]*)**.** For any maximal operator T, the resolvent operator associated with T, for any $\rho > 0$, is defined as

$$J_T(u) = (I + \rho T)^{-1}(u), \quad \forall u \in H.$$

It is well known that an operator T is maximal monotone if and only if its resolvent operator J_T is defined everywhere. It is single valued and nonexpansive.

If $\varphi(.)$ is a proper, convex and lower semicontinuous function, then its subdifferential $\partial\varphi(.)$ is a maximal monotone operator. In this case, we can define the resolvent operator

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \forall u \in H$$

associated with the subdifferential $\partial\varphi(.)$. The resolvent operator J_φ has the following useful characterization.

Lemma 2.3. *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \forall v \in H$$

if and only if

$$u = J_\varphi(z),$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

It is well known that the resolvent operator J_φ is nonexpansive, that is,

$$\|J_\varphi u - J_\varphi v\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Definition 2.4. An operator $T : H \to H$ is said to be:

(i) *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \quad u, v \in H;$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \quad u, v \in H.$$

We note that if T satisfies (i) and (ii), then it is clear that $\alpha \leq \beta$.

3. The resolvent operator method

In this section, we suggest and analyze some new approximation schemes for solving the extended general mixed variational inequality (2.5). One can prove that the extended general mixed variational inequality problem (2.5) is equivalent to the fixed point problem by invoking Lemma 2.3. However, for the sake of completeness and to convey an idea of the technique, we include the proof.

Lemma 3.1. *The function $u \in H$ is a solution of the general mixed variational inequality* (2.5) *if and only if $u \in H$ satisfies the relation*

$$h(u) = J_\varphi[g(u) - \rho Tu], \tag{3.1}$$

where J_φ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of problem (2.5). Then

$$\langle h(u) - (g(u) - \rho Tu), g(v) - h(u)\rangle + \rho\varphi(g(v)) - \rho\varphi(h(u)) \geq 0, \quad \forall v \in H,$$

which we know, using Lemma 2.3, to be equivalent to

$$h(u) = J_\varphi[g(u) - \rho Tu],$$

the required result. □

Lemma 3.1 implies that the extended general mixed variational inequality problem (2.5) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical points of view.

We rewrite the relation (3.1) in the following form:

$$F(u) \equiv u - h(u) + J_\varphi[g(u) - \rho Tu], \tag{3.2}$$

which is used to study the existence of a solution of the extended general mixed variational inequality (2.5) and this is the main motivation for our next result.

Theorem 3.1. *Let the operators $T, g, h : H \longrightarrow H$ be strongly monotone with constants $\alpha > 0, \sigma > 0, \nu$ and Lipschitz continuous with constants $\beta > 0, \delta > 0, \mu$, respectively. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)},\ k = \sqrt{1 - 2\sigma + \delta^2}, < 1, \tag{3.3}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\nu + \mu^2}, \tag{3.4}$$

then there exists a solution of problem (2.5).

Proof. From Lemma 3.1, it follows that problems (2.1) and (3.1) are equivalent. Thus it is enough to show that the map $F(u)$ defined by (3.2) has a fixed point. For all $u \neq v \in H$, we have

$$\begin{aligned}\|F(u) - F(v)\| &= \|u - v - (h(u) - h(v))\| + \|J_\varphi[g(u) - \rho Tu] - J_\varphi[g(v) - \rho Tv]\| \\ &\leq \|u - v - (h(u) - h(v))\| + \|g(u) - g(v) - \rho(Tu - Tv)\| \\ &\leq \|u - v - (h(u) - h(v))\| + \|u - v - (g(u) - g(v))\| + \|u - v - \rho(Tu - Tv)\|. \end{aligned} \tag{3.5}$$

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}\|u - v - \rho(Tu - Tv)\|^2 &\leq \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \end{aligned} \tag{3.6}$$

In a similar way, we have

$$\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2\sigma + \delta^2)\|u - v\|^2 \tag{3.7}$$

$$\|u - v - (h(u) - h(v))\|^2 \leq (1 - 2\nu + \mu^2)\|u - v\|^2. \tag{3.8}$$

From (3.5)–(3.8), we have

$$\begin{aligned}\|u_{n+1} - u\| &\leq \{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\nu + \mu^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u - v\| \\ &= (k + t(\rho))\|u - v\|, \\ &= \theta\|u_n - u\|, \end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2} \tag{3.9}$$

and

$$\theta = k + t(\rho). \tag{3.10}$$

From (3.3), it follows that $\theta < 1$. Thus the mapping $F(u)$ defined by (3.2) is a contraction mapping and consequently has a fixed point belonging to H satisfying the general mixed variational inequality (2.5). □

Using the fixed point formulation (3.1), we suggest and analyze the following iterative method for solving the extended general mixed variational inequalities (2.5).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by using the iterative schemes

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{u_n - h(u_n) + J_\varphi[g(u_n) - \rho Tu_n]\}, \quad n = 0, 1, \ldots \tag{3.11}$$

which is known as the Mann iteration process for solving th general variational inequalities, where $0 \leq \alpha_n \leq 1$.

Note that if $g = I$, then Algorithm 3.1 reduces to the following iterative method for solving the mixed variational inequalities (2.6).

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by using the iterative schemes

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{u_n - h(u_n) + J_\varphi[u_n - \rho Tu_n]\}, \quad n = 0, 1, \ldots,$$

where $0 \leq \alpha_n \leq 1$.

If the function φ is a an indicator function of a closed convex set K, then $J_\varphi \equiv P_K$, the projection of H onto the convex set K. Consequently, Algorithm 3.1 reduces to the following iterative method for solving the extended general variational inequality (2.8), which is mainly due to Noor [4–9].

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by using the iterative scheme

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - h(u_n) + P_K[g(u_n) - \rho T u_n]\}, \quad n = 0, 1, \ldots,$$

where $0 \le \alpha_n \le 1$.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation for our next result.

Theorem 3.2. *Let the operators T, g, h satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds and $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.1 converges to a solution $u \in H$ satisfying the extended general mixed variational inequality (2.5).*

Proof. From Theorem 3.1, it follows that there exists a solution $u \in H$ of problem (2.5). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - h(u) + J_\varphi[g(u) - \rho T u]\}, \tag{3.12}$$

where $0 \le \alpha_n \le 1$ is a constant.

From (3.1) and (3.12), we have

$$\begin{aligned} \|u_{n+1} - u\| &= \|(1-\alpha_n)(u_n - u)\| + \alpha_n\|u_n - u - (h(u_n) - h(u))\| + \|J_\varphi[g(u_n) - \rho T u_n] - J_\varphi[g(u) - \rho T u]\| \\ &\le (1-\alpha_n)\|u_n - u\| + \alpha_n\|u_n - u - (h(u_n) - h(u))\| \\ &\quad + \alpha_n\|u_n - u - (g(u_n) - g(u))\| + \alpha_n\|u_n - u - \rho(T u_n - T u)\|. \end{aligned} \tag{3.13}$$

From (3.6)–(3.9) and (3.13), we have

$$\begin{aligned} \|u_{n+1} - u\| &\le (1-\alpha_n)\|u_n - u\| + \alpha_n\sqrt{1 - 2\nu + \mu^2}\|u_n - u\| + \alpha_n\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_n - u\| \\ &= (1-\alpha_n)\|u_n - u\| + \alpha_n(k + t(\rho))\|u_n - u\|, \\ &= (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|u_n - u\|, \end{aligned} \tag{3.14}$$

where $t(\rho)$ and θ are defined by (3.9) and (3.10) respectively.

From (3.3), it follows that $\theta < 1$. Thus

$$\begin{aligned} \|u_{n+1} - u\| &\le (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|u_n - u\| \\ &= [1 - (1-\theta)\alpha_n]\|u_n - u\| \\ &\le \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\|u_0 - u\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\longrightarrow\infty}\left\{\prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\right\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

4. The resolvent equation technique

In this section, we use the resolvent equation technique to suggest and analyze an iterative method for solving the extended general mixed variational inequality (2.5).

We consider the problem of solving the resolvent equations. To be more precise, suppose that $R_\varphi = I - gh^{-1}J_\varphi$ where I is the identity operator and g is a given nonlinear operator. For given nonlinear operators T, g, h, we consider the problem of finding $z \in H$ such that

$$Th^{-1}J_\varphi z + \rho^{-1}R_\varphi z = 0, \tag{4.1}$$

which is called the extended general resolvent equation. We note that if $g = h = I$, then one can obtain the original resolvent equations, which are mainly due to Noor [24]. It has been shown that the resolvent equations have played an important and significant role in developing several numerical techniques for solving extended general mixed variational inequalities and related optimization problems; see [24–32,4–6,8,33,36].

If the proper, convex and lower semicontinuous function φ is an indicator function of a closed convex set K, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. Consequently, the extended general resolvent equation problem (4.1) is equivalent to that of finding $z \in H$ such that

$$Th^{-1}P_K z + \rho^{-1}Q_K z = 0,$$

which are called the extended general Wiener–Hopf equations (see [7,8]), where $Q_K = I - gh^{-1}P_K$. For $g = h = I$, one can obtain the original Wiener–Hopf equations of Shi [40]. For the applications, sensitivity analysis and numerical methods for solving Wiener–Hopf equations, see [23,2,24–32,4–8,34,39,40] and the references therein.

For this purpose, we need the following result.

Lemma 4.1. *The solution $u \in H$ satisfies the extended general mixed variational inequality (2.5) if and only if $z \in H$ is a solution of the extended general resolvent equation (4.1), where*

$$h(u) = J_\varphi z \tag{4.2}$$

$$z = g(u) - \rho Tu, \quad \rho > 0, \text{ a constant.} \tag{4.3}$$

Proof. Let $u \in H$ be a solution of (2.5). Then, from Lemma 3.1, we have

$$h(u) = J_\varphi[g(u) - \rho Tu]. \tag{4.4}$$

Suppose that

$$z = g(u) - \rho Tu. \tag{4.5}$$

Then

$$h(u) = J_\varphi z. \tag{4.6}$$

Combining (4.6), (4.4) and (4.5), we have

$$z = g(u) - \rho Tu = gh^{-1}J_\varphi z - \rho Th^{-1}J_\varphi z,$$

from which it follows that $z \in H$ is a solution of the extended general resolvent equation (4.1), the required result. □

Lemma 4.1 implies that the general mixed variational inequalities (2.5) and the extended general resolvent equation (4.1) are equivalent. We use this equivalent formulation to suggest a number of iterative methods for solving the extended general mixed variational inequalities (2.5).

I. Using (4.2), the resolvent equation (4.1) can be rewritten in the form

$$R_\varphi z = -\rho Th^{-1}J_\varphi z,$$

which implies that

$$z = gh^{-1}J_\varphi z - \rho Th^{-1}J_\varphi z = g(u) - \rho Tu.$$

This fixed point formulation enables us to suggest the following iterative method for solving problem (2.5).

Algorithm 4.1. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by using the iterative schemes

$$h(u_n) = J_\varphi z_n, \tag{4.7}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{g(u_n) - \rho Tu_n\}, \quad n = 0, 1, \ldots, \tag{4.8}$$

where $0 \le \alpha_n \le 1$ for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. By an appropriate and suitable rearrangement of the terms and using (4.2), the resolvent equation (4.1) can be written as

$$\begin{aligned} z &= gh^{-1}J_\varphi z - \rho Th^{-1}J_\varphi z + (1 - \rho^{-1})R_\varphi z \\ &= g(u) - \rho Tu + (1 - \rho^{-1})J_\varphi z, \end{aligned}$$

which is another fixed point formulation. Using this fixed point formulation, we can suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by using the iterative schemes

$$h(u_n) = J_\varphi z_n$$

$$z_{n+1} = g(u_n) - \rho Tu_n + (1 - \rho^{-1})R_\varphi z_n, \quad n = 0, 1, \ldots.$$

III. If T is linear and T^{-1} exists, then the general resolvent Eq. (4.1) can be written as

$$z = \left(I - \rho^{-1}gT^{-1}\right) R_\varphi z.$$

This fixed point formulation allows us to suggest the following iterative method for solving the extended general mixed variational inequalities (2.5).

Algorithm 4.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by using the iterative scheme

$$z_{n+1} = \left(I - \rho^{-1} g T^{-1}\right) R_\varphi z_n, \quad n = 0, 1, \dots.$$

For $g = I$, the identity operator, Algorithm 4.1-Algorithm 4.3 are due to Noor [18]. In brief, by appropriate and suitable rearrangements of the terms of the general resolvent equation (4.1), one can suggest and analyze a number of iterative methods for solving the general mixed variational inequalities (2.1) and (2.5) and related optimization problems. For the investigation of such resolvent iterative methods and the verification of their numerical efficiency, further research efforts are needed.

We now consider the convergence analysis of Algorithm 4.1. In a similar way, one can study the convergence analysis of Algorithms 4.2 and 4.3.

Theorem 4.1. *Let the operators T, g satisfy all the assumptions of Theorem 3.1. If the condition of (3.3) holds, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the resolvent form Eq. (2.13) strongly.*

Proof. Let $u \in H$ be a solution of (2.5). Then, using Lemma 4.1, we have

$$z = (1 - \alpha_n)z + \alpha_n\{g(u) - \rho Tu\}, \tag{4.9}$$

where $0 \le \alpha_n \le 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From (4.8), (4.9), (3.7) and (3.6), we have

$$\begin{aligned} \|z_{n+1} - z\| &\le (1 - \alpha_n)\|z_n - z\| + \alpha_n\|g(u_n) - g(u) - \rho(Tu_n - Tu)\| \\ &\le (1 - \alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\| + \alpha_n\|u_n - u - \rho(Tu_n - Tu)\| \\ &\le (1 - \alpha_n)\|z_n - z\| + \alpha_n\{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\}\|u_n - u\|. \end{aligned} \tag{4.10}$$

Also from (4.6) and (4.9) and the nonexpansivity of the resolvent operator J_φ, we have

$$\begin{aligned} \|u_n - u\| &\le \|u_n - u - (h(u_n) - h(u))\| + \|J_\varphi z_n - J_\varphi z\| \\ &\le \sqrt{1 - 2\nu + \mu^2}\|u_n - u\| + \|z_n - z\| \end{aligned}$$

which implies that

$$\|u_n - u\| \le \frac{1}{1 - \sqrt{1 - 2\nu + \mu^2}}\|z_n - z\|. \tag{4.11}$$

Combining (4.11) and (4.10), we have

$$\|z_{n+1} - z\| \le (1 - \alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\|, \tag{4.12}$$

where

$$\theta_1 = \frac{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}}{1 - \sqrt{1 - 2\nu + \mu^2}}.$$

Using (3.3), we see that $\theta < 1$ and consequently

$$\begin{aligned} \|z_{n+1} - z\| &\le (1 - \alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\| \\ &= [1 - (1 - \theta)\alpha_n]\|z_n - z\| \\ &\le \prod_{i=0}^{n}[1 - (1 - \theta)\alpha_i]\|z_0 - z\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\longrightarrow\infty}\left\{\prod_{i=0}^{n}[1 - (1 - \theta)\alpha_i]\right\} = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result. □

5. Conclusion

In this paper, we have introduced and considered a new class of extended general variational inequalities involving four different operators. We have established the equivalence between the extended general variational inequalities, fixed point problems and resolvent equations. This equivalence is used to suggest and analyze some iterative methods for solving the extended general variational inequalities. Several special cases are also discussed. For some recent advances in this field, see Noor et al. [35,36]. Using the techniques of Liu and Cao [14] and Liu and Yang [15], one can develop a neural network for solving the extended general mixed variational inequalities, which is another direction for future research work. We hope that the ideas and techniques of this paper may stimulate further research in this field.

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